Normality of cut polytopes of graphs is a minor closed property

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Abstract

Sturmfels–Sullivant conjectured that the cut polytope of a graph is normal if and only if the graph has no K_5 minor. In the present paper, it is proved that the normality of cut polytopes of graphs is a minor closed property. By using this result, we have large classes of normal cut polytopes. Moreover, it turns out that, in order to study the conjecture, it is enough to consider 4-connected plane triangulations.

1 Introduction

Let G be a graph on the vertices $[n] := \{1, 2, \ldots, n\}$ and edges E without loops or multiple edges. Let $S \subset [n]$. Then the cut semimetric on G induced by S is the 0/1 vector $\delta_G(S)$ in $\mathbb{R}^E$ defined by

$$\delta_G(S)_{ij} = \begin{cases} 1 & \text{if } |S \cap \{i,j\}| = 1 \\ 0 & \text{otherwise} \end{cases}$$

where $ij \in E$. Let $A_G = \{\mathbf{a}_1, \ldots, \mathbf{a}_N\} = \{\delta_G(S) \mid S \subset [n]\} \subset \mathbb{Z}^E$ where $N = 2^{n-1}$. The *cut polytope* $\mathrm{Cut}^\square(G)$ of G is the convex hull of A_G. Let

$$\begin{aligned}
X_G &:= \left\{ \begin{pmatrix} \mathbf{a}_1 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} \mathbf{a}_N \\ 1 \end{pmatrix} \right\} \subset \mathbb{Z}^{E+1}, \\
\mathbb{Z}(X_G) &:= \left\{ \sum_{i=1}^N z_i \begin{pmatrix} \mathbf{a}_i \\ 1 \end{pmatrix} \;\middle|\; z_i \in \mathbb{Z} \right\} \subset \mathbb{Z}^{E+1}, \\
\mathbb{Q}_+(X_G) &:= \left\{ \sum_{i=1}^N q_i \begin{pmatrix} \mathbf{a}_i \\ 1 \end{pmatrix} \;\middle|\; 0 \le q_i \in \mathbb{Q} \right\} \subset \mathbb{Q}^{E+1}, \\
\mathbb{Z}_+(X_G) &:= \left\{ \sum_{i=1}^N z_i \begin{pmatrix} \mathbf{a}_i \\ 1 \end{pmatrix} \;\middle|\; 0 \le z_i \in \mathbb{Z} \right\} \subset \mathbb{Z}^{E+1}.
\end{aligned}$$

Then $\mathbb{Z}_+(X_G) \subset \mathbb{Z}(X_G) \cap \mathbb{Q}_+(X_G)$ holds in general. The cut polytope $\mathrm{Cut}^\square(G)$ is called *normal* if we have $\mathbb{Z}_+(X_G) = \mathbb{Z}(X_G) \cap \mathbb{Q}_+(X_G)$.

1.1 A conjecture on normal cut polytopes

Let $K[\mathbf{t}, s] = K[t_1, \ldots, t_E, s]$ be the polynomial ring in $E + 1$ variables over a field K and let $K[\mathbf{q}] = K[q_1, \ldots, q_N]$ the polynomial ring in $N(= 2^{n-1})$ variables over K. For each nonnegative integer vector $\alpha = (\alpha_1, \ldots, \alpha_E) \in \mathbb{Z}^E$, we set $\mathbf{t}^\alpha = t_1^{\alpha_1} \cdots t_E^{\alpha_E}$. Then the *toric cut ideal* I_G of a graph G is the kernel of homomorphism $\pi : K[\mathbf{q}] \longrightarrow K[\mathbf{t}, s]$ defined by $\pi(q_i) = \mathbf{t}^{\mathbf{a}_i} s$. Sturmfels–Sullivant [StSu, Conjecture 3.7] conjectured that $K[\mathbf{q}]/I_G$ is normal if and only if G has no K_5 minor. Since it is known (e.g., [Stu, Proposition 13.5]) that $K[\mathbf{q}]/I_G$ is normal if and only if $\mathbb{Z}_+(X_G) = \mathbb{Z}(X_G) \cap \mathbb{Q}_+(X_G)$ holds, their conjecture is formulated as follows:

Conjecture 1.1. The cut polytope $\mathrm{Cut}^\square(G)$ is normal if and only if G has no K_5 minor.

If $\mathrm{Cut}^\square(G)$ is normal and G' is obtained from G by contracting an edge, then $\mathrm{Cut}^\square(G')$ is normal ([StSu, Lemma 3.2 (2)]). Note that, if a graph G has K_m as a minor, then that minor can be realized by a sequence of edge contraction only. As stated in [StSu], the "only if" part is true since $\mathrm{Cut}^\square(K_5)$ is not normal. On the other hand, the "if" part is true for the following classes of graphs:

- graphs with ≤ 6 vertices (by a direct computation [StSu] together with [StSu, Theorem 1.2])
- graphs having no induced cycle of length ≥ 5 (by [Sul1, Theorem 3.2])
- "ring graphs" (Note that ring graphs have no K_4 minor. See [NaPe]).

1.2 Hilbert bases of cut polytopes

In order to avoid confusion, we must introduce "nonhomogeneous" version of this problem on cut polytopes. The following sets are studied in, e.g., [FuGo, Lau]:

$$\begin{aligned}
\mathbb{Z}(A_G) &:= \left\{ \sum_{i=1}^{N} z_i \mathbf{a}_i \;\middle|\; z_i \in \mathbb{Z} \right\} \subset \mathbb{Z}^E \\
\mathbb{Q}_+(A_G) &:= \left\{ \sum_{i=1}^{N} q_i \mathbf{a}_i \;\middle|\; 0 \leq q_i \in \mathbb{Q} \right\} \subset \mathbb{Q}^E \\
\mathbb{Z}_+(A_G) &:= \left\{ \sum_{i=1}^{N} z_i \mathbf{a}_i \;\middle|\; 0 \leq z_i \in \mathbb{Z} \right\} \subset \mathbb{Z}^E
\end{aligned}$$

If $\mathbb{Z}_+(A_G) = \mathbb{Z}(A_G) \cap \mathbb{Q}_+(A_G)$ holds, then A_G is called a *Hilbert basis*. It is known that $\mathbb{Z}_+(A_G) = \mathbb{Z}(A_G) \cap \mathbb{Q}_+(A_G)$ holds if one of the following holds:

- G has no K_5 minor ([FuGo, Corollary 1.3]);
- G is $K_6 \setminus e$ or its subgraph ([Lau, Theorem 1.1]).

Moreover, $\mathbb{Z}_+(A_G) \neq \mathbb{Z}(A_G) \cap \mathbb{Q}_+(A_G)$ holds if

- G has K_6 minor ([Lau, Proposition 1.2]).

On the other hand, it is known that the class of graphs G satisfying $\mathbb{Z}_+(A_G) = \mathbb{Z}(A_G) \cap \mathbb{Q}_+(A_G)$ is closed under

- contraction minors ([Lau, Proposition 2.1]);
- clique sums ([Lau, Proposition 2.7]);
- edge deletions satisfying some conditions ([Lau, Proposition 2.3]).

Hence it is natural to have the following conjecture.

Conjecture 1.2. *Let G be a connected graph. Then $\mathbb{Z}_+(A_G) = \mathbb{Z}(A_G) \cap \mathbb{Q}_+(A_G)$ if and only if G has no K_6 minor.*

The relation between our problem and this problem is as follows:

Proposition 1.3. *If $\mathbb{Z}_+(X_G) = \mathbb{Z}(X_G) \cap \mathbb{Q}_+(X_G)$ holds, then we have $\mathbb{Z}_+(A_G) = \mathbb{Z}(A_G) \cap \mathbb{Q}_+(A_G)$.*

Proof. Suppose that $\mathbb{Z}_+(X_G) = \mathbb{Z}(X_G) \cap \mathbb{Q}_+(X_G)$ holds. Let $\mathbf{x} \in \mathbb{Z}(A_G) \cap \mathbb{Q}_+(A_G)$. Since $(0, \ldots, 0, 1) \in \mathbb{Z}(X_G)$, there exists an integer α such that

$$\begin{pmatrix} \mathbf{x} \\ \alpha \end{pmatrix} \in \mathbb{Z}(X_G) \cap \mathbb{Q}_+(X_G) = \mathbb{Z}_+(X_G).$$

Thus $\mathbf{x} \in \mathbb{Z}_+(A_G)$ as desired. □

Remark 1.4. The graph K_5 is a counterexample of the converse of Proposition 1.3.

1.3 Main results

The main purpose of the present paper is to prove that the set of graphs G such that $\mathrm{Cut}^{\square}(G)$ is normal is minor closed (Corollary 2.4). Thanks to Corollary 2.4, we have large classes of normal cut polytopes (Theorem 3.3, Corollary 3.6 and Theorem 3.8). In addition, in Section 4, we will show that, in order to study Conjecture 1.1, it is enough to consider 4-connected plane triangulations.

Since the converse of Proposition 1.3 is not true in general (Remark 1.4), we cannot apply the results on Hilbert bases to our problem *directly.* However there are a lot of useful idea in [Lau]. For example, the idea of the proof of Theorem 2.3 comes from that of [Lau, Proposition 2.3] and the proof of Theorem 3.2 is similar to that of [Lau, Proposition 2.7].

2 Deletion of an edge

Since the origin belongs to A_G, we have $(0, \ldots, 0, 1) \in X_G$. Hence it follows from [Lau, p.258] that, for $\mathbf{x} \in \mathbb{Z}^E$ and $\alpha \in \mathbb{Z}$,

$$\begin{pmatrix} \mathbf{x} \\ \alpha \end{pmatrix} \in \mathbb{Z}(X_G) \iff \sum_{e \in C} x_e \equiv 0 \pmod 2 \tag{1}$$

for each cycle C of G. From now on, we always assume that G has no K_5 minor. Then the following Proposition is known.

Proposition 2.1 ([BaMa])**.** *Let G be a graph without K_5 minor. Then $\mathrm{Cut}^{\square}(G)$ is the solution set of the following linear inequalities:*

$$0 \leq x_e \leq 1, \ e \in E$$

$$\sum_{e \in F} x_e - \sum_{e \in C \setminus F} x_e \leq |F| - 1$$

where C ranges over the induced cycles of G and F ranges over the odd subsets of C.

Thanks to Proposition 2.1, we have the following:

Corollary 2.2. *Let G be a graph without K_5 minor. For a vector $\mathbf{x} \in \mathbb{Q}^E$ and a nonnegative integer α, $\begin{pmatrix} \mathbf{x} \\ \alpha \end{pmatrix} \in \mathbb{Q}_+(X_G)$ if and only if*

$$0 \leq x_e \leq \alpha, \ e \in E$$

$$\sum_{e \in F} x_e - \sum_{e \in C \setminus F} x_e \leq \alpha \ (|F| - 1)$$

where C ranges over the induced cycles of G and F ranges over the odd subsets of C.

Proof. It follows from the following fact:

$$\frac{1}{\alpha}\mathbf{x} \in \mathrm{Cut}^{\square}(G) \iff \begin{pmatrix} \mathbf{x} \\ \alpha \end{pmatrix} \in \mathbb{Q}_+(X_G)$$

for $0 < \alpha \in \mathbb{Z}$ and $\mathbf{x} \in \mathbb{Q}^E$. $\square$

By using the equation (1) together with Corollary 2.2, we have the following.

Theorem 2.3. *Let G be a graph. If $\mathrm{Cut}^{\square}(G)$ is normal, then $\mathrm{Cut}^{\square}(G \setminus e_0)$ is normal for any edge e_0 of G.*

Proof. The idea of the proof is obtained from that of [Lau, Proposition 2.3]. Let $G' = G \setminus e_0$. Note that G and G' have no K_5 minor. Let $A_{G'} = \{\mathbf{a}_1, \ldots, \mathbf{a}_N\}$ and

$$\begin{pmatrix} \mathbf{x} \\ \alpha \end{pmatrix} = \sum_{i=1}^{N} q_i \begin{pmatrix} \mathbf{a}_i \\ 1 \end{pmatrix} \in \mathbb{Z}(X_{G'}) \cap \mathbb{Q}_+(X_{G'})$$

where $0 < \alpha \in \mathbb{Z}$ and $0 \le q_i \in \mathbb{Q}$ for $1 \le i \le N$. Since $\mathrm{Cut}^{\square}(G)$ is normal, it is enough to show that there exists a nonnegative integer γ such that

$$\begin{pmatrix} \gamma \\ \mathbf{x} \\ \alpha \end{pmatrix} \in \ \mathbb{Z}(X_G) \cap \mathbb{Q}_+(X_G) \ = \ \mathbb{Z}_+(X_G).$$

Let $\mathbf{x}' = \begin{pmatrix} \gamma \\ \mathbf{x} \end{pmatrix}$ where $\gamma \in \mathbb{Q}$. Thanks to Corollary 2.2, $\begin{pmatrix} \mathbf{x}' \\ \alpha \end{pmatrix} \in \mathbb{Q}_+(X_G)$ if and only if

$$0 \le \gamma \le \alpha \tag{2}$$

$$\sum_{e \in F} x'_e - \sum_{e \in C \setminus F} x'_e \le \alpha \ (|F| - 1) \tag{3}$$

where C ranges over the induced cycles of G with $e_0 \in C$ and F ranges over the odd subsets of C. Then the equations (2) and (3) have a solution γ. In fact,

$$\sum_{i=1}^{N} q_i \begin{pmatrix} \delta_i \\ \mathbf{a}_i \\ 1 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{N} q_i \delta_i \\ \mathbf{x} \\ \alpha \end{pmatrix} \in \mathbb{Q}_+(X_G),$$

where $A_G = \left\{ \begin{pmatrix} \delta_1 \\ \mathbf{a}_1 \end{pmatrix}, \ldots, \begin{pmatrix} \delta_N \\ \mathbf{a}_N \end{pmatrix} \right\}$. Let

$$\begin{aligned} x_{\max} &= \max_{(C,\ F)\ |\ e_0 \in C \setminus F} \left(\sum_{e \in F} x'_e - \sum_{e \in C \setminus F,\ e \neq e_0} x'_e - \alpha \ (|F| - 1) \right) \in \mathbb{Z}, \\ x_{\min} &= \min_{(C,\ F)\ |\ e_0 \in F} \left(- \sum_{e \in F,\ e \neq e_0} x'_e + \sum_{e \in C \setminus F} x'_e + \alpha \ (|F| - 1) \right) \in \mathbb{Z}. \end{aligned}$$

Note that $|F| - 1$ is even. By (2) and (3) above, we have

$$\begin{pmatrix} \gamma \\ \mathbf{x} \\ \alpha \end{pmatrix} \in \mathbb{Q}_+(X_G) \iff \max(0, x_{\max}) \le \gamma \le \min(\alpha, x_{\min}). \tag{4}$$

On the other hand, let C be an arbitrary cycle of G containing e_0. Then by (1),

$$\begin{pmatrix} \gamma \\ \mathbf{x} \\ \alpha \end{pmatrix} \in \mathbb{Z}(X_G) \iff \gamma \equiv \sum_{e \in C,\ e \neq e_0} x'_e \pmod 2. \tag{5}$$

If $\max(0, x_{\max}) < \min(\alpha, x_{\min})$, then $\max(0, x_{\max}) + 1 \leq \min(\alpha, x_{\min})$ and hence either $\gamma = \max(0, x_{\max})$ or $\gamma = \max(0, x_{\max}) + 1$ satisfies the conditions (4) and (5). Suppose that $\max(0, x_{\max}) = \min(\alpha, x_{\min})$. Let $\gamma = \max(0, x_{\max}) = \min(\alpha, x_{\min}) \in \mathbb{Z}$. Since $0 < \alpha$, at least one of $\gamma = x_{\max}$ or $\gamma = x_{\min}$ holds. If $\gamma = x_{\max}$, then there exists a cycle C of G containing e_0 such that

$$\begin{aligned} \gamma &= \sum_{e \in F} x'_e - \sum_{e \in C \setminus F,\ e \neq e_0} x'_e - \alpha\ (|F| - 1) \\ &\equiv \sum_{e \in C,\ e \neq e_0} x'_e \pmod 2. \end{aligned}$$

Similarly, if $\gamma = x_{\min}$, then there exists a cycle C of G containing e_0 such that

$$\begin{aligned} \gamma &= -\sum_{e \in F,\ e \neq e_0} x'_e + \sum_{e \in C \setminus F} x'_e + \alpha\ (|F| - 1) \\ &\equiv \sum_{e \in C,\ e \neq e_0} x'_e \pmod 2. \end{aligned}$$

In both cases, γ satisfies the conditions (4) and (5). Thus we have

$$\begin{pmatrix} \gamma \\ \mathbf{x} \\ \alpha \end{pmatrix} \in \mathbb{Z}(X_G) \cap \mathbb{Q}_+(X_G) = \mathbb{Z}_+(X_G)$$

and hence $\begin{pmatrix} \mathbf{x} \\ \alpha \end{pmatrix} \in \mathbb{Z}_+(X_{G'})$ as desired. $\square$

It is known [StSu, Lemma 3.2 (2)] that, if $\mathrm{Cut}^{\square}(G)$ is normal and G' is obtained from G by contracting an edge, then $\mathrm{Cut}^{\square}(G')$ is normal. Thus, we have the following:

Corollary 2.4. *The set of graphs G such that $\mathrm{Cut}^{\square}(G)$ is normal is minor closed.*

3 Clique sums and normality

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs such that $V_1 \cap V_2$ is a clique of both graphs. The new graph $G = G_1 \sharp G_2$ with the vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$ is called the *clique sum* of G_1 and G_2 along $V_1 \cap V_2$. If the cardinality of $V_1 \cap V_2$ is $k + 1$, this operation is called a *k-sum* of the graphs.

Proposition 3.1 ([StSu]). *Let $G = G_1 \sharp G_2$ be a 0, 1 or 2-sum of G_1 and G_2. Then the set of generators (or Gröbner bases) of the toric ideal I_G of $\mathrm{Cut}^{\square}(G)$ consists of that of I_{G_1} and I_{G_2} together with some quadratic binomials.*

It turns out that this holds even for normality.

Theorem 3.2. *Let $G = G_1 \sharp G_2$ be a 0, 1 or 2-sum of G_1 and G_2. Then the cut polytope of G is normal if and only if the cut polytope of G_i is normal for $i = 1, 2$.*

Proof. This is similar to the proof of [Lau, Proposition 2.7].

Since G_1 and G_2 are induced subgraphs of G, the "only if" part follows from [StSu, Lemma 3.2 (1)].

Suppose that the cut polytope of G_i is normal for $i = 1, 2$. Let $\{i_1, \ldots, i_k\}$ $(1 \le k \le 3)$ denote the common vertices of G_1 and G_2. It is easy to see that we can express A_G as

$$A_G = \{\delta_G(S) \mid i_1 \in S \subset [n]\} \subset \mathbb{Z}^E. \tag{6}$$

Case 1. $k = 3$

By (6), we have $A_G = A_G^{++} \cup A_G^{+-} \cup A_G^{-+} \cup A_G^{--}$ where

$$\begin{aligned}
A_G^{++} &= \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z}_0 \end{pmatrix} \,\middle|\, \begin{pmatrix} \mathbf{x} \\ \mathbf{z}_0 \end{pmatrix} \in A_{G_1}^{++}, \begin{pmatrix} \mathbf{y} \\ \mathbf{z}_0 \end{pmatrix} \in A_{G_2}^{++} \right\}, \quad \mathbf{z}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
A_G^{+-} &= \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z}_1 \end{pmatrix} \,\middle|\, \begin{pmatrix} \mathbf{x} \\ \mathbf{z}_1 \end{pmatrix} \in A_{G_1}^{+-}, \begin{pmatrix} \mathbf{y} \\ \mathbf{z}_1 \end{pmatrix} \in A_{G_2}^{+-} \right\}, \quad \mathbf{z}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\
A_G^{-+} &= \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z}_2 \end{pmatrix} \,\middle|\, \begin{pmatrix} \mathbf{x} \\ \mathbf{z}_2 \end{pmatrix} \in A_{G_1}^{-+}, \begin{pmatrix} \mathbf{y} \\ \mathbf{z}_2 \end{pmatrix} \in A_{G_2}^{-+} \right\}, \quad \mathbf{z}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\
A_G^{--} &= \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z}_3 \end{pmatrix} \,\middle|\, \begin{pmatrix} \mathbf{x} \\ \mathbf{z}_3 \end{pmatrix} \in A_{G_1}^{--}, \begin{pmatrix} \mathbf{y} \\ \mathbf{z}_3 \end{pmatrix} \in A_{G_2}^{--} \right\}, \quad \mathbf{z}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
A_{G_i}^{++} &= \{\delta_{G_i}(S) \mid i_1, i_2, i_3 \in S \subset [n_i]\} \subset \mathbb{Z}^{E_i} \\
A_{G_i}^{+-} &= \{\delta_{G_i}(S) \mid i_1, i_2 \in S \subset [n_i], \quad i_3 \notin S\} \subset \mathbb{Z}^{E_i} \\
A_{G_i}^{-+} &= \{\delta_{G_i}(S) \mid i_1, i_3 \in S \subset [n_i], \quad i_2 \notin S\} \subset \mathbb{Z}^{E_i} \\
A_{G_i}^{--} &= \{\delta_{G_i}(S) \mid i_1 \in S \subset [n_i], \quad i_2, i_3 \notin S\} \subset \mathbb{Z}^{E_i}.
\end{aligned}$$

Let $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ p \\ q \\ r \\ \alpha \end{pmatrix} \in \mathbb{Z}(X_G) \cap \mathbb{Q}_+(X_G)$ for a positive integer α. Then we have

$$\begin{pmatrix} \mathbf{x} \\ p \\ q \\ r \\ \alpha \end{pmatrix} \in \mathbb{Z}(X_{G_1}) \cap \mathbb{Q}_+(X_{G_1}) = \mathbb{Z}_+(X_{G_1}), \quad \begin{pmatrix} \mathbf{y} \\ p \\ q \\ r \\ \alpha \end{pmatrix} \in \mathbb{Z}(X_{G_2}) \cap \mathbb{Q}_+(X_{G_2}) = \mathbb{Z}_+(X_{G_2}).$$

Hence

$$\begin{pmatrix} \mathbf{x} \\ p \\ q \\ r \\ \alpha \end{pmatrix} = \begin{pmatrix} \mathbf{x}^{(1)} \\ \mathbf{z}_{k_1} \\ 1 \end{pmatrix} + \begin{pmatrix} \mathbf{x}^{(2)} \\ \mathbf{z}_{k_2} \\ 1 \end{pmatrix} + \cdots + \begin{pmatrix} \mathbf{x}^{(\alpha)} \\ \mathbf{z}_{k_\alpha} \\ 1 \end{pmatrix} \text{ where } \begin{pmatrix} \mathbf{x}^{(i)} \\ \mathbf{z}_{k_i} \end{pmatrix} \in A_{G_1} \quad (7)$$

$$\begin{pmatrix} \mathbf{y} \\ p \\ q \\ r \\ \alpha \end{pmatrix} = \begin{pmatrix} \mathbf{y}^{(1)} \\ \mathbf{z}_{k'_1} \\ 1 \end{pmatrix} + \begin{pmatrix} \mathbf{y}^{(2)} \\ \mathbf{z}_{k'_2} \\ 1 \end{pmatrix} + \cdots + \begin{pmatrix} \mathbf{y}^{(\alpha)} \\ \mathbf{z}_{k'_\alpha} \\ 1 \end{pmatrix} \text{ where } \begin{pmatrix} \mathbf{y}^{(j)} \\ \mathbf{z}_{k'_j} \end{pmatrix} \in A_{G_2}. \quad (8)$$

Let ξ_i (resp. ξ'_i) denote the number of $\mathbf{z}_i$ appearing in (7) (resp. (8)) for each $i = 0, 1, 2, 3$. Then we have $p = \xi_2+\xi_3 = \xi'_2+\xi'_3$, $q = \xi_1+\xi_3 = \xi'_1+\xi'_3$, $r = \xi_1+\xi_2 = \xi'_1+\xi'_2$, and $\alpha = \sum_{i=0}^{4} \xi_i = \sum_{i=0}^{4} \xi'_i$. Hence $\xi_i = \xi'_i$ for all $i = 0, 1, 2, 3$. Thus, by changing the numbering, we have

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ p \\ q \\ r \\ \alpha \end{pmatrix} = \begin{pmatrix} \mathbf{x}^{(1)} \\ \mathbf{y}^{(1)} \\ \mathbf{z}_{k_1} \\ 1 \end{pmatrix} + \begin{pmatrix} \mathbf{x}^{(2)} \\ \mathbf{y}^{(2)} \\ \mathbf{z}_{k_2} \\ 1 \end{pmatrix} + \cdots + \begin{pmatrix} \mathbf{x}^{(\alpha)} \\ \mathbf{y}^{(\alpha)} \\ \mathbf{z}_{k_\alpha} \\ 1 \end{pmatrix} \in \mathbb{Z}_+(X_G).$$

Case 2. $k = 1, 2$

By (6), if $k = 1$, then $A_G = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \,\middle|\, \mathbf{x} \in A_{G_1},\ \mathbf{y} \in A_{G_2} \right\}$ and if $k = 2$, then we

have $A_G = A_G^+ \cup A_G^-$ where

$$\begin{aligned}
A_G^+ &= \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ 0 \end{pmatrix} \;\middle|\; \begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix} \in A_{G_1}^+, \begin{pmatrix} \mathbf{y} \\ 0 \end{pmatrix} \in A_{G_2}^+ \right\} \\
A_G^- &= \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ 1 \end{pmatrix} \;\middle|\; \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \in A_{G_1}^-, \begin{pmatrix} \mathbf{y} \\ 1 \end{pmatrix} \in A_{G_2}^- \right\} \\
A_{G_i}^+ &= \{\delta_{G_i}(S) \mid i_1, i_2 \in S \subset [n_i]\} \subset \mathbb{Z}^{E_i} \\
A_{G_i}^- &= \{\delta_{G_i}(S) \mid i_1 \in S \subset [n_i], i_2 \notin S\} \subset \mathbb{Z}^{E_i}.
\end{aligned}$$

In both cases, the desired conclusion follows from the similar (and simpler) argument in Case 1. □

A graph $G = (V, E)$ is called *edge-maximal without $\mathcal{H}$ minor*, if G has no $\mathcal{H}$ minor but any graph $G' = (V, E')$ with $E' = E \cup \{e\}$ and $e \notin E$ has $\mathcal{H}$ minor.

Let G be a graph with vertex set $V = [n] = \{1, \ldots, n\}$ and edge set E. The *suspension* of the graph G is the new graph $\widehat{G}$ whose vertex set equals $[n+1] = V \cup \{n+1\}$ and whose edge set equals $E \cup \{\{i, n+1\} \mid i \in V\}$. A cut ideal $I_{\widehat{G}}$ corresponds to the toric ideal arising from the binary graph model of G.

Theorem 3.3. *Let G be a graph. Then $\mathrm{Cut}^{\square}(\widehat{G})$ is normal if and only if G has no K_4 minor.*

Proof. If G has K_4 minor, then $\widehat{G}$ has K_5 minor. Hence $\mathrm{Cut}^{\square}(\widehat{G})$ is not normal.

It is known [Die2, Proposition 7.3.1] that a graph with at least three vertices is edge-maximal without K_4 minor if and only if it is 1 sum of K_3's. Hence, if G is edge-maximal without K_4 minor, then $\widehat{G}$ is 2 sums of K_4's. Since the cut polytope of K_4 is normal, $\mathrm{Cut}^{\square}(\widehat{G})$ is normal by Theorem 3.2. Thus for any subgraph G' of G, $\mathrm{Cut}^{\square}(\widehat{G'})$ is normal by Theorem 2.3. □

Remark 3.4. One of the referees pointed out that Theorem 3.3 implies the main result of [Sul2].

Example 3.5. The cut polytope of a wheel graph $W_n = \widehat{C_n}$ is normal since the cycle C_n has no K_4 minor.

By considering the subgraph of the graphs appearing in Theorem 3.3, we have

Corollary 3.6. *If G has a vertex v such that the induced subgraph of G on $V \setminus \{v\}$ has no K_4 minor, then $\mathrm{Cut}^{\square}(G)$ is normal.*

Example 3.7. Let G be a graph with ≤ 5 vertices. Then the cut polytope of G is normal if and only if $G \neq K_5$.

Theorem 3.8. *Let G be a graph with no $K_5 \setminus e$ minor. Then* $\mathrm{Cut}^{\square}(G)$ *is normal.*

Proof. It is known [Die1, p.180] that, if G is edge-maximal graph without $K_5 \setminus e$ minor, then G is obtained by 1-sum of the graphs K_3, $K_{3,3}$, W_n, and the prism $C_3 \times K_2$. Since the cut polytope of all of them are normal, $\mathrm{Cut}^{\square}(G)$ is normal by Theorem 3.2. By Theorem 2.3, the cut polytope of any subgraph of G is normal. □

4 Sturmfels–Sullivant Conjecture

Although Conjecture 1.1 is still open, the following is known [Die1, p.181] in graph theory.

Proposition 4.1. *Let G be an edge-maximal graph without K_5 minor. If G has at least 3 vertices, then G is 1 or 2 sum of K_3, K_4, 4-connected plane triangulations and the graph V_8.*

The cut polytopes of K_3 and K_4 are normal. Moreover,

Example 4.2. Let V_8 be the graph with the edge set

$$\{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,6\},\{6,7\},\{7,8\},\{1,8\},\{1,5\},\{2,6\},\{3,7\},\{4,8\}\}.$$

Since V_8 has an induced cycle of length 5, $\mathrm{Cut}^{\square}(V_8)$ is not compressed by [Sul1, Theorem 3.2]. It follows from Corollary 3.6 that the cut polytope of any proper minor of V_8 is normal. By the software `Normaliz` [BrIc], we can check that $\mathrm{Cut}^{\square}(V_8)$ is normal.

Thus, in order to prove Conjecture 1.1, it is enough to prove one of the following conjectures:

Conjecture 4.3. *The cut polytope* $\mathrm{Cut}^{\square}(G)$ *is normal if G is a 4-connected plane triangulation.*

Conjecture 4.4. *The cut polytope* $\mathrm{Cut}^{\square}(G)$ *is normal if G is a grid graph.*

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